

## SOLUTION OF THE PROBLEM OF THE MONTH, SEPTEMBER 2019

Consider the integer sequence  $\{x_n\}_{n \geq 0}$  given by  $x_0 = 0; x_1 = 1$  and

$$x_n = 4x_{n-1} - x_{n-2}; \text{ for all } n \geq 2:$$

The first few terms of this sequence are

$0; 1; 4; 15; 56; 209; 780; 2911; 10864; 40545; 151316; 564719; 2107560; 7865521; 29354524; \dots$

Find the smallest  $n \geq 2$  such that  $x_n$  is a prime number, or prove that such an  $n$  does not exist.

*Solution.* It turns out that for every  $n \geq 2$ , the term  $x_n$  is composite. We will need the following statement which can be easily proved by induction.

*For every  $n \geq 2$  we have*

$$(1) \quad x_{n+1}^2 - 4x_n x_{n+1} + x_n^2 = 1:$$

Assume that  $x_{n+1} = p$ , where  $p$  is a prime  $\geq 3$ . Then, the above equality can be written as  $p^2 - 4px_n + x_n^2 = 1$  or  $x_n^2 - 4px_n + p^2 - 1 = 0$ . Regard this as a quadratic equation in  $x_n$ .

Since  $x_n$  is an integer, the discriminant must be a perfect square, that is,  $(4p)^2 - 4(p^2 - 1) = 16p^2 - 4p^2 + 4 = 12p^2 + 4 = 4(3p^2 + 1)$  is a perfect square. This implies that  $3p^2 + 1$  is a perfect square. Let  $3p^2 + 1 = q^2$ , then  $q^2 - 3p^2 = 1$ . This is a Pell equation. The fundamental solution is  $(q, p) = (2, 1)$ . The solutions are given by  $(q_n, p_n) = (2 + 3n, 1 + 3n)$ . For  $n \geq 1$ ,  $p_n = 1 + 3n$  is not a prime number. Therefore, no  $n \geq 2$  exists such that  $x_n$  is a prime number.