SOLUTION OF THE PROBLEM OF THE MONTH, SEPTEMBER 2019

Consider the integer sequence $\{x_n\}_{n \geq 0}$ given by $x_0 = 0; x_1 = 1$ and

$$x_n = 4x_{n-1} - x_{n-2}; \quad \text{for all } n \geq 2.$$

The first few terms of this sequence are

$$0; 1; 4; 15; 56; 209; 780; 2911; 10864; 40545; 151316; 564719; 2107560; 7865521; 29354524; \ldots$$

Find the smallest $n \geq 2$ such that x_n is a prime number, or prove that such an n does not exist.

Solution. It turns out that for every $n \geq 2$, the term x_n is composite. We will need the following statement which can be easily proved by induction.

For every $n \geq 2$ we have

(1) $$x_{n+1}^2 - 4x_n x_{n+1} + x_n^2 = 1.$$

Assume that $x_{n+1} = p$, where p is a prime ≥ 3. Then, the above equality can be written as $p^2 - 4px_n + x_n^2 = 1$ or $x_n^2 - 4px_n + p^2 - 1 = 0$. Regard this as a quadratic equation in x_n.

Since x_n is an integer, the discriminant must be a perfect square, that is, (4 d 1 0 0 (5 0 4)] T 687 Td [(1 0 8 6) - 28 7 971 Tf (4 1 2 . 527 - 1 0 - 385Td